ME 323: FLUID MECHANICS-II

Dr. A.B.M. Toufique Hasan

Professor

Department of Mechanical Engineering

Bangladesh University of Engineering & Technology (BUET), Dhaka

Lecture-02

09/09/2024

Compressible Flow

toufiquehasan.buet.ac.bd
toufiquehasan@me.buet.ac.bd

Isentropic flow

For compressible flows, change in thermodynamic property '***entropy***' is important. From the definition of entropy, s; (review course of Engineering Thermodynamics (ME 203))

$$\Delta s = s_2 - s_1 = \int \frac{\delta Q}{T}\bigg|_{\text{reversible}}$$

where δQ represents the differential heat transfer.

For an **ideal gas (calorically perfect)** with constant specific heats; the change of entropy becomes as:

$$\Delta s = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

If the process is adiabatic (no heat transfer) and reversible, the **entropy change is zero** and then the flow is said to be ***isentropic***. In this case

$$\Delta s = 0 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

The isentropic approximation is common in compressible flow theory.

Isentropic flow

$$\Delta s = 0 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

$$\Rightarrow R \ln \frac{p_2}{p_1} = c_p \ln \frac{T_2}{T_1}$$

$$\Rightarrow \ln \frac{p_2}{p_1} = \frac{c_p}{R} \ln \frac{T_2}{T_1}$$

$$\Rightarrow \ln \frac{p_2}{p_1} = \frac{c_p}{c_p - c_v} \ln \frac{T_2}{T_1} \qquad ; c_p - c_v = R$$

$$\Rightarrow \ln \frac{p_2}{p_1} = \frac{k}{k-1} \ln \frac{T_2}{T_1} \qquad ; k = \frac{c_p}{c_v}$$

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}}$$

⇐ **Power-law relation**

Relates the ratio of pressure at two different states with temperature at those states in an isentropic flow.

Thermodynamic property relations for isentropic flows.

Isentropic flow

From ideal gas EOS (Equation of states)

$$p = \rho RT$$

$$\therefore \frac{p_2}{p_1} = \frac{\rho_2 RT_2}{\rho_1 RT_1}$$

$$\Rightarrow \frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} \times \frac{T_2}{T_1}$$

$$\Rightarrow \frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} \times \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} \qquad ; \because \frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}}$$

$$\Rightarrow \left(\frac{p_2}{p_1}\right)^{1-\frac{k-1}{k}} = \frac{\rho_2}{\rho_1}$$

$$\Rightarrow \frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{k}$$

⇐ **Power-law relation**

Relates the ratio of pressure at two different states with density at those states in an isentropic flow.

Thermodynamic property relations for isentropic flows

Flow Energy Equation

Recall the **Euler differential equation*** (from RTT & differential control volume analysis: ME 321):

$$\frac{dp}{\rho} + d\left(\frac{V^2}{2}\right) + gdz = 0$$

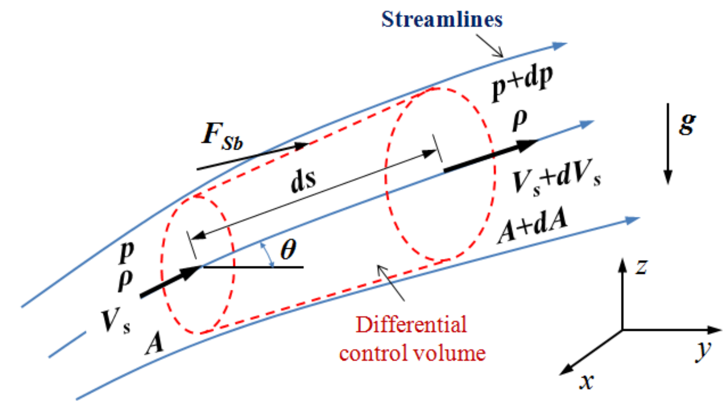


Fig. Differential control volume for momentum analysis of flow through a stream tube

Integrate the Euler equation along a streamline:

$$\int \frac{dp}{\rho} + \int d\left(\frac{V^2}{2}\right) + \int gdz = \text{Constant}$$

$$\Rightarrow \int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{Constant}$$

for incompressible flow (ρ = constant), Bernoulli equation:

$$\Rightarrow \frac{p}{\rho} + \frac{V^2}{2} + gz = \text{Constant}$$

$$\Rightarrow \frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{Constant}$$

for compressible flow ($\rho \neq$ constant), rather $\rho = f(p)$.

???

*Ref. ME 321 (Fluid Mechanics-I)

Flow Energy Equation

From isentropic flow relation for ideal gas:

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^k$$

$$\rightarrow \frac{p_2}{\rho_2{}^k} = \frac{p_1}{\rho_1{}^k} \equiv \frac{p}{\rho^k} = \text{Constant} \equiv G_1$$

$$\Rightarrow \frac{p}{\rho^k} = G_1$$

$$\Rightarrow \frac{p^{\frac{1}{k}}}{\rho} = G_2 \quad (\text{another constant, } G_2 = G_1^{\frac{1}{k}})$$

$$\rightarrow \frac{1}{\rho} = G_2 p^{-\frac{1}{k}}$$

Then: $$\int \frac{dp}{\rho} = \int \frac{1}{\rho} dp = \int \left(G_2 p^{-\frac{1}{k}}\right) dp = G_2 \int p^{-\frac{1}{k}} dp$$

$$\Rightarrow \int \frac{dp}{\rho} = G_2 \frac{p^{-\frac{1}{k}+1}}{-\frac{1}{k}+1} + G_3$$

$$\Rightarrow \int \frac{dp}{\rho} = G_2 \frac{p^{\frac{k-1}{k}}}{\frac{k-1}{k}} + G_3 \quad (\text{another constant of integration, } G_3)$$

Flow Energy Equation

$$\Rightarrow \int \frac{dp}{\rho} = \frac{p^{\frac{1}{k}}}{\rho} \frac{p^{\frac{k-1}{k}}}{\frac{k-1}{k}} + G_3$$

$$\frac{p^{\frac{1}{k}}}{\rho} = G_2$$

$$= \left(\frac{k}{k-1}\right) \frac{p^{\frac{1}{k}+\frac{k-1}{k}}}{\rho} + G_3$$

$$\Rightarrow \int \frac{dp}{\rho} = \left(\frac{k}{k-1}\right) \frac{p}{\rho} + G_3$$

So, the integrated form of the Euler equation along a streamline for compressible isentropic flow comes as:

$$\Rightarrow \int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{Constant}$$

$$\Rightarrow \left(\frac{k}{k-1}\right) \frac{p}{\rho} + \frac{V^2}{2} + gz = \text{Constant}$$

Compressible form of Bernoulli Eq:

Notice the difference

Incompressible form of Bernoulli Eq:

$$\Rightarrow \frac{p}{\rho} + \frac{V^2}{2} + gz = \text{Constant}$$

Flow Energy Equation

In general, the potential head is insignificant in flows with compressibility; then-

$$(gz)_{\text{compressible}} \approx 0$$

$$\Rightarrow \left(\frac{k}{k-1}\right)\frac{p}{\rho} + \frac{V^2}{2} = \text{Constant}$$

Flow Energy equation for compressible flow.
i.e. compressible form of Bernoulli Equation

for air (k = 1.4), this factor accounts a magnitude of 3.5;
whereas this factor is 1.0 for incompressible flow for any fluid.

$$\phi\frac{p}{\rho} + \frac{V^2}{2} = \text{Constant}$$

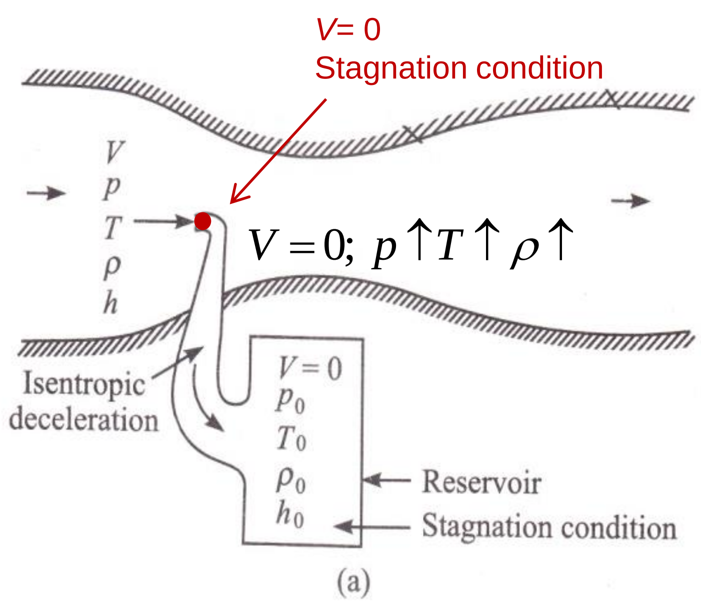
$$\phi = 1.0 \quad \text{for incompressible flow}$$

$$\phi = \frac{k}{k-1} \quad \text{for compressible flow}$$

Stagnation state (total state)

The **stagnation condition** (total condition) is a reference state for compressible flow. This condition is extremely useful in the analysis of compressible flow. *These are the quantities with subscript zero* (0).

The **stagnation state** of a flowing fluid is defined by the state attained by the flowing fluid when it is decelerated to **zero velocity through isentropic process**. Alternatively, it can be defined as the static state from which a fluid must be accelerated isentopically in order to attain the actual state for a given flow.



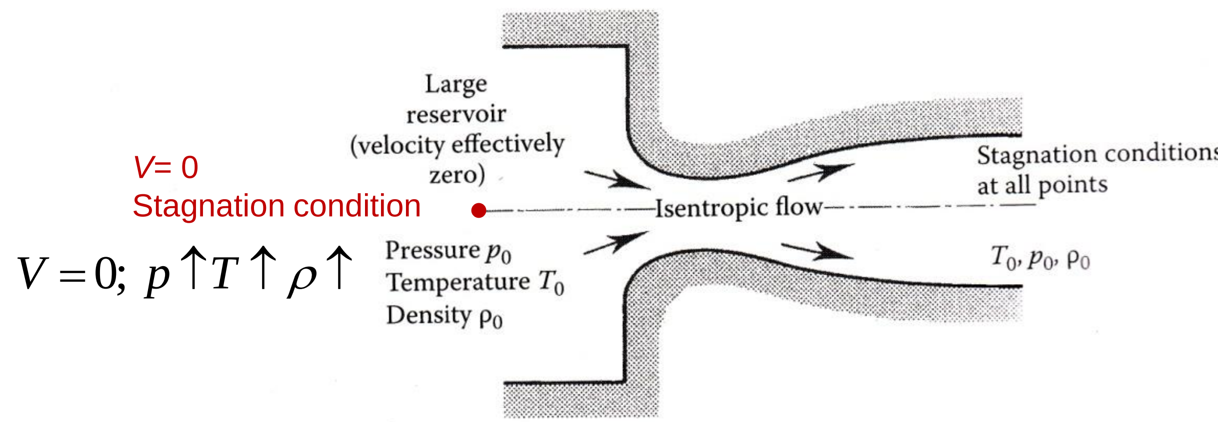


FIGURE 4.4
Stagnation conditions in an isentropic flow.

Stagnation (total) properties are-

- Total pressure, p_0
- Total temperature, T_0
- Total density, ρ_0
- Total enthalpy, h_0

Stagnation state (total state)

Pressure cooker

Aerosol spray

Spray paint

CNG cylinder in a vehicle

Accidents due to CNG cylinder bursting

Stagnation temperature (total temperature)

Compressible form of Bernoulli Equation:

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1}+\frac{V_1^2}{2}=\left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2}+\frac{V_2^2}{2}$$

①: any local point (V≠0)

②: stagnation point (V=0)

$$\Rightarrow\left(\frac{k}{k-1}\right)\frac{p}{\rho}+\frac{V^2}{2}=\left(\frac{k}{k-1}\right)\frac{p_0}{\rho_0}+\frac{V_0^2}{2}$$

$$\Rightarrow\left(\frac{k}{k-1}\right)RT+\frac{V^2}{2}=\left(\frac{k}{k-1}\right)RT_0+\frac{0^2}{2} \qquad ;\; p=\rho RT$$

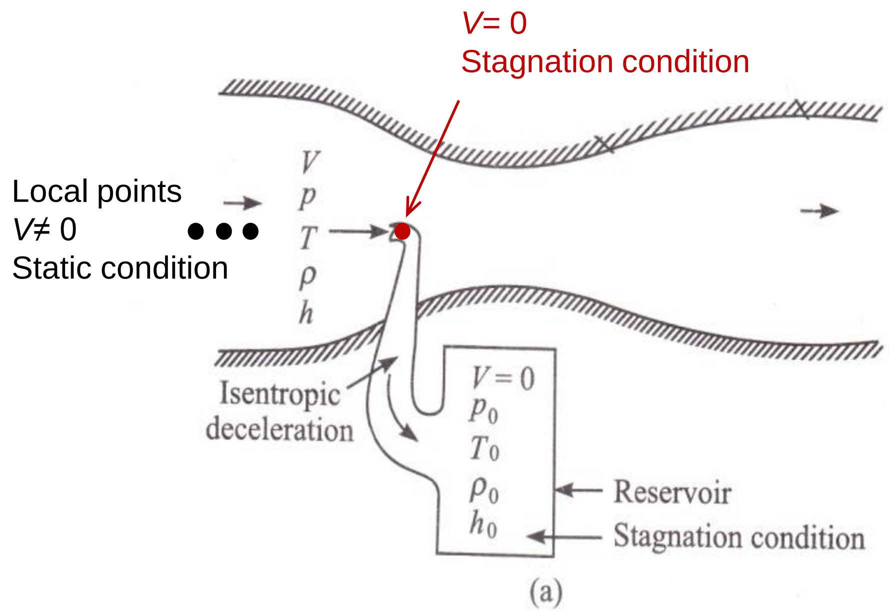
$$\Rightarrow\left(\frac{k}{k-1}\right)RT+\frac{V^2}{2}=\left(\frac{k}{k-1}\right)RT_0$$

$$\Rightarrow 1+\frac{V^2}{2}\left(\frac{k-1}{kRT}\right)=\left(\frac{k}{k-1}\right)RT_0\times\left(\frac{k-1}{kRT}\right) \qquad ;\times\left(\frac{k-1}{kRT}\right)$$

$$\Rightarrow 1+\frac{V^2}{2}\left(\frac{k-1}{a^2}\right)=\frac{T_0}{T} \qquad ;\; \text{speed of sound, } a=\sqrt{kRT}$$

$$\Rightarrow 1+\frac{k-1}{2}M^2=\frac{T_0}{T} \qquad ;\; \text{Mach number, } M=\frac{V}{a}$$

$$\Rightarrow \frac{T_0}{T}=1+\frac{k-1}{2}M^2$$



$$\frac{T_0}{T}=1+\frac{k-1}{2}M^2$$

Relation between stagnation and static temperatures

Stagnation pressure (total pressure)

Thermodynamic property relations for isentropic flow-

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}}$$

①: any local point (V≠0), $p_1 = p$

②: stagnation point (V=0), $p_2 = p_0$

$$\rightarrow \frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{k}{k-1}}$$

$$\because \frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$$

$$\rightarrow \frac{p_0}{p} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}}$$

Relation between stagnation and static pressures

Flow Mach number can be conveniently calculated from:

$$\rightarrow M = \sqrt{\frac{2}{k-1}\left[\left(\frac{p_0}{p}\right)^{\frac{k-1}{k}} - 1\right]}$$

Stagnation density (total density)

Thermodynamic property relations for isentropic flow-

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^k$$

①: any local point (V≠0), $\rho_1 = \rho$

②: stagnation point (V=0), $\rho_2 = \rho_0$

$$\rightarrow \frac{p_0}{p} = \left(\frac{\rho_0}{\rho}\right)^k$$

$$\rightarrow \frac{\rho_0}{\rho} = \left(\frac{p_0}{p}\right)^{\frac{1}{k}}$$

$$\rightarrow \frac{\rho_0}{\rho} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{1}{k-1}}$$

$$\because \frac{p_0}{p} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}}$$

Relation between stagnation and static densities

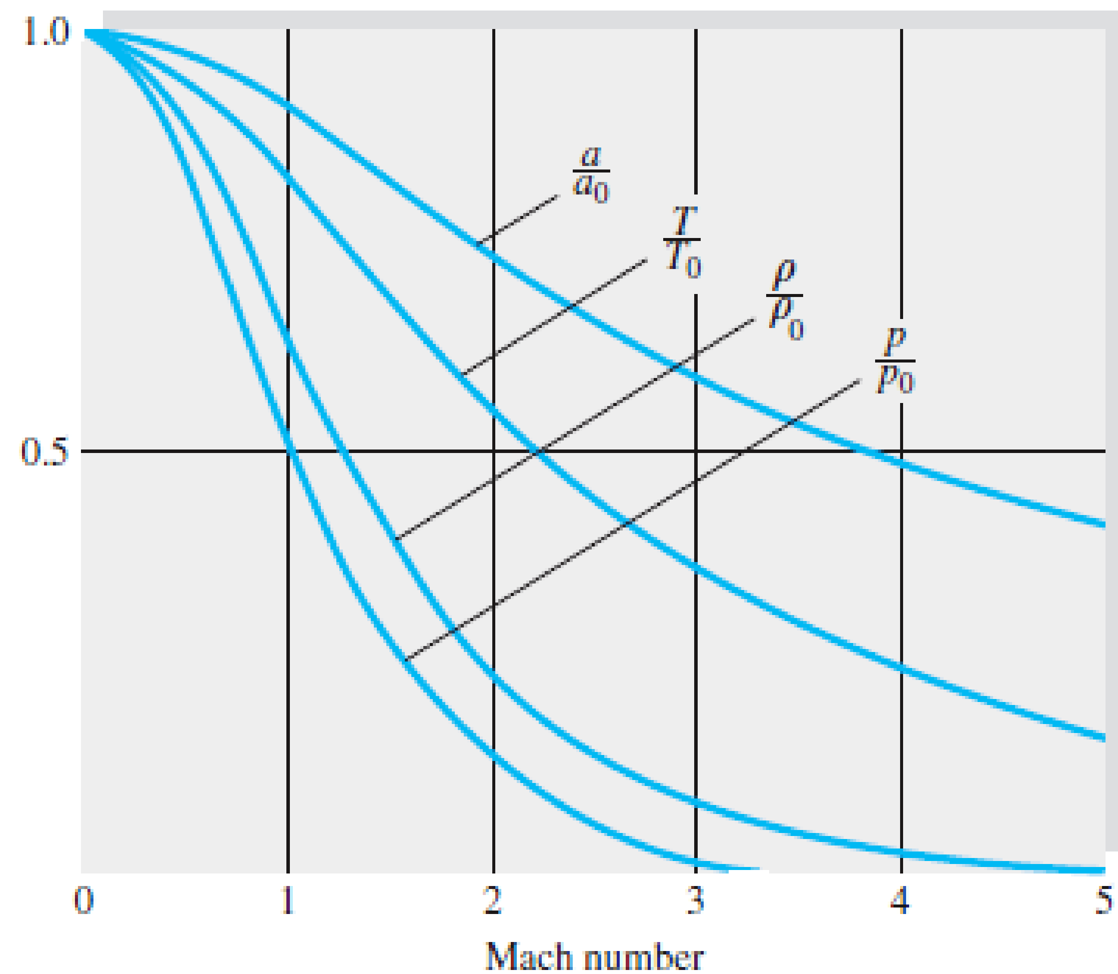


Fig. 9.3 Adiabatic (T/T_0 and a/a_0) and isentropic (p/p_0 and ρ/ρ_0) properties versus Mach number for $k = 1.4$.

are all gas flows compressible flows??

Not all gas flows are compressible flows, neither are all compressible flows gas flows

Table: Variation of density with *M*

M	ρ_0/ρ	$\Delta\rho$
0.1	1.005	0.5%
0.2	1.02	2%
0.3	1.04	4%
0.4	1.08	8%
0.5	1.13	13%
1.0	1.58	58%
2.0	4.35	335%

$$\Rightarrow \frac{\rho_0}{\rho} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{1}{k-1}}$$

For air k =1.4

General High speed/bullet train Shinkansen (Japan), TGV (France)

At low speeds, less than Mach number of about 0.3 (~100 m/s, 360 km/hr at STP), gas flows may be treated as incompressible flows since the density variations caused by the flow **are less than 5% which is insignificant in engineering sense.**